

UPPER AND LOWER BOUNDS FOR KRONECKER CONSTANTS OF THREE-ELEMENT SETS OF INTEGERS

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ABSTRACT. Various upper and lower bounds are provided for the (angular) Kronecker constants of sets of integers. Some examples are provided where the bounds are attained. It is proved that $5/16$ bounds the angular Kronecker constants of 3-element sets of positive integers. However, numerous examples suggest that the minimum upper bound is $1/4$ for 3-element sets of positive integers.

1. INTRODUCTION

A subset S of the dual of a compact, abelian group G is called an ε -Kronecker set if for every continuous function f mapping S into \mathbb{T} , the set of complex numbers of modulo 1, there exists $x \in G$ such that

$$|\gamma(x) - f(\gamma)| < \varepsilon \text{ for all } \gamma \in S.$$

The infimum of such ε is called the Kronecker constant, $\kappa(S)$.

We continue with the notation of [Hare and Ramsey], and define an angular Kronecker constant $\alpha(S) \in [0, 1/2]$ such that

$$\kappa(S) = \left| e^{2\pi i \alpha(S)} - 1 \right|$$

In this note, the terminology, notations and results of [Hare and Ramsey] will be used.

This note is a supplement to [Hare and Ramsey], giving some bounds for angular Kronecker constants of three-element sets of integers:

- Theorem 1 uses the full machinery of [Hare and Ramsey] to produce both upper and lower bounds for 3-element sets of non-zero, relatively prime integers with distinct absolute values.
- Theorem 1 and two results of [Hare and Ramsey] are used to prove that $\alpha(S) \leq 5/16$ for all S consisting of non-zero integers with distinct absolute values. See Theorem 2. Examples suggest that one should be able to lower this upper bound to $1/4$ for $5/16$.

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2. BOUNDING ESTIMATES FOR (MOST) THREE ELEMENT SETS OF INTEGERS

[Hare and Ramsey] provides an easy upper bound (better than $1/2$) for any finite set S of integers that does not contain 0:

$$\alpha(S) \leq \frac{1}{2} - \frac{1}{2d}$$

where d is the size of S .

When $d = 2$, this gives $\alpha(S) \leq 1/4$ and this trivial upper bound is sharp if and only if $S = \{-n, n\}$. For $d = 3$, the bound is $1/3$ and [Hare and Ramsey] shows that this is sharp when $S = \{-n, n, 2n\}$. A consequence of our next theorem is that the angular Kronecker constant is strictly less than $1/3$ for all other three element sets (that exclude 0).

Theorem 1. *Suppose $|n_1|, |n_2|, |n_3|$ are distinct and $\gcd(n_1, n_2, n_3) = 1$. Assume $m = \gcd(n_2, n_3)$ and that $(1/m, 0)$ and $(r/n_3, m/n_3)$ generate the lattice \mathcal{K} for some $r > 0$. We have the following bounds on angular Kronecker constants:*

$$\alpha\{n_1, n_2, n_3\} \geq \frac{|n_3|}{2(r(|n_2| + |n_3|) + m(|n_1| + |n_3|))}$$

and

$$\alpha\{n_1, n_2, n_3\} \leq \frac{E_1 (2|n_1||n_2| + |n_3|(|n_1| + |n_2|))}{|n_3|(|n_1| + |n_2|)}$$

where

$$E_1 = \max \left(\frac{m}{2(|n_2| + |n_3|)}, \frac{r}{2(|n_1| + |n_3|)}, \frac{|n_3| + 2rm}{2(r(|n_2| + |n_3|) + m(|n_1| + |n_3|))} \right).$$

Proof. Temporarily fix (x, y) and put $\beta = x - ry/m$. For an integer t , put $\Delta_t = y - tm/n_3$. With this notation, the angular Kronecker constant is the least constant E such that for each β there are integers s, t such that

$$\begin{aligned} (2, 3) &: \frac{|n_3|}{|n_2| + |n_3|} |\Delta_t| \leq E \\ (1, 3) &: \frac{|n_3|}{|n_1| + |n_3|} \left| \beta - \frac{s}{m} + \Delta_t \frac{r}{m} \right| \leq E \\ (1, 2) &: \frac{|n_2(\beta - \frac{s}{m} + \Delta_t \frac{r}{m}) - n_1 \Delta_t|}{|n_1| + |n_2|} \leq E. \end{aligned}$$

The requirements (1, 3) and (2, 3) can be satisfied if and only for each β there are integers s, t such that $\Delta_t \in J_1(s, E) \cap J_2(s, E)$ where J_1, J_2 are the intervals

$$\begin{aligned} J_1(s, E) &= \left[-E \left(\frac{|n_2| + |n_3|}{|n_3|} \right), E \left(\frac{|n_2| + |n_3|}{|n_3|} \right) \right] \text{ and} \\ J_2(s, E) &= \left[\frac{m}{r} \left(-E \left(\frac{|n_1| + |n_3|}{|n_3|} \right) - \beta + \frac{s}{m} \right), \frac{m}{r} \left(E \left(\frac{|n_1| + |n_3|}{|n_3|} \right) - \beta + \frac{s}{m} \right) \right]. \end{aligned}$$

If there is some choice of β such that $J_1(s, E) \cap J_2(s, E)$ is empty for all $s \in \mathbb{Z}$, then clearly (1, 3) and (2, 3) cannot be simultaneously satisfied for that choice of E . Hence it is necessary that for each β there be an integer s with the right end of $J_1(s, E) \geq$ left end of $J_2(s, E)$ and the right end of $J_2(s, E) \geq$ left end of $J_1(s, E)$. This implies that if we let

$$c(E) = \frac{E}{|n_3|} \left(\frac{r}{m} (|n_2| + |n_3|) + |n_1| + |n_3| \right),$$

then a necessary condition on E is that for each β there is an integer s satisfying

$$-c(E) \leq -\beta + \frac{s}{m} \leq c(E).$$

However, if $\beta = 1/2m$ and $c(E) < 1/2m$ then this inequality cannot hold for any integer s . Thus a necessary condition for E (a lower bound on the Kronecker constant) is that E must satisfy

$$\frac{E}{|n_3|} \left(\frac{r}{m}(|n_2| + |n_3|) + |n_1| + |n_3| \right) \geq \frac{1}{2m}$$

and this gives the lower bound stated in the theorem.

Now put $E_1 = E_0 + \lambda$, where E_0 is the lower bound and $\lambda > 0$ is to be determined. Then $c(E_1) \geq 1/2m$, so for any x, y there will be an integer s such that $J_1(s, E_1) \cap J_2(s, E_1)$ is non-empty. If, in addition, the length of the overlap of the two intervals is at least $m/|n_3|$, then we can be sure that for any choice of y , there will be an integer t with $\Delta_t = y - tm/n_3 \in J_1(s, E_1) \cap J_2(s, E_1)$.

The length of the overlap will be either

$$(1) \quad 2E_1 \left(\frac{|n_2| + |n_3|}{|n_3|} \right) \text{ if } J_1 \subseteq J_2$$

$$(2) \quad 2E_1 \left(\frac{|n_1| + |n_3|}{|n_3|} \right) \frac{m}{r} \text{ if } J_2 \subseteq J_1$$

or

$$(3) \quad E_1 \left(\frac{|n_2| + |n_3|}{|n_3|} \right) + \left(E_1 \frac{|n_1| + |n_3|}{|n_3|} - \left| -\beta + \frac{s}{m} \right| \right) \frac{m}{r} \text{ otherwise.}$$

Obviously, if

$$E_1 \geq \max \left(\frac{m}{2(|n_2| + |n_3|)}, \frac{r}{2(|n_1| + |n_3|)} \right)$$

then both (1) and (2) will be at least $m/|n_3|$.

Now consider (3). Note that the choice of E_0 ensures that for any β there is an integer s with

$$-\left| -\beta + \frac{s}{m} \right| \geq -\frac{1}{2m} = \frac{-E_0}{2m} \left(\frac{r}{m}(|n_2| + |n_3|) + |n_1| + |n_3| \right).$$

Thus

$$E_1 \left(\frac{|n_2| + |n_3|}{|n_3|} \right) + \left(E_1 \frac{|n_1| + |n_3|}{|n_3|} - \left| -\beta + \frac{s}{m} \right| \right) \frac{m}{r} \geq \frac{\lambda(r(|n_2| + |n_3|) + m(|n_1| + |n_3|))}{r|n_3|}.$$

and therefore we can satisfy the inequality (3) $\geq m/|n_3|$ if we choose

$$\lambda \geq \frac{rm}{r(|n_2| + |n_3|) + m(|n_1| + |n_3|)},$$

that is,

$$E_1 \geq \frac{|n_3| + 2rm}{2(r(|n_2| + |n_3|) + m(|n_1| + |n_3|))}.$$

The construction ensures that if we take $E = E_1$, then both inequalities (1, 3) and (2, 3) can be simultaneously satisfied for each x, y , with a suitable choice of integers s, t .

But then,

$$|\Delta_t| \leq E_1 \left(\frac{|n_2| + |n_3|}{|n_3|} \right) \text{ and } \left| \beta - \frac{s}{m} + \Delta_t \frac{r}{m} \right| \leq E_1 \left(\frac{|n_1| + |n_3|}{|n_3|} \right)$$

and therefore

$$\begin{aligned} (1, 2) &\leq \frac{|n_2| E_1 \left(\frac{|n_1| + |n_3|}{|n_3|} \right) + |n_1| E_1 \left(\frac{|n_2| + |n_3|}{|n_3|} \right)}{|n_1| + |n_2|} \\ &= \frac{E_1 (2 |n_1| |n_2| + |n_3| (|n_1| + |n_2|))}{|n_3| (|n_1| + |n_2|)}. \end{aligned}$$

This verifies the claimed upper bound on the angular Kronecker constant. \square

To apply this result to show that the Kronecker constant is less than $1/3$ for any three element set not containing 0, other than the sets $\{-n, n, 2n\}$, it is convenient to first record some preliminary calculations. The arguments are elementary and we will only give the main idea for each.

Lemma 1. *Assume $0 < |n_1| < n_2 < n_3$ and that integers $r, m \geq 1$.*

(i) *If $E_1 \leq n_3/4(|n_1| + n_3)$, then*

$$\frac{E_1 (2 |n_1| n_2 + n_3 (|n_1| + n_2))}{n_3 (|n_1| + n_2)} \leq \frac{5}{16}.$$

(ii) *If $r + m \geq 5$, then*

$$\frac{2 |n_1| n_2 + n_3 (|n_1| + n_2)}{(|n_1| + n_2) (r(n_2 + n_3) + m(|n_1| + n_3))} \leq \frac{5}{16}.$$

(iii) *If $(r, m) \neq (1, 1)$, then*

$$\frac{(2 |n_1| n_2 + n_3 (|n_1| + n_2)) (2 + \min(r, m))}{4(|n_1| + n_2) (r(n_2 + n_3) + m(|n_1| + n_3))} \leq \frac{5}{16}.$$

Proof. (i) Consider the cases $|n_1| \geq n_2/2$ and $|n_1| \leq n_2/2$ separately. In the first case use the fact that $4n_1^2 \geq 2|n_1|n_2$; in the second, use the inequality $n_3(|n_1| + n_2) \geq 3|n_1|n_2$.

(ii) The key idea here is the inequality $(|n_1| + n_2) (rn_2 + m|n_1|) \geq 7|n_1|n_2$.

(iii) Here it is convenient to put $s = \min(r, m)$ and write $r + m = 2s + l$ for $l \geq 0$. Then consider the two possibilities: $s \geq 2$, $l \geq 0$ and $s = 1$, $l \geq 1$. \square

Lemma 2. *Assume $0 < |n_1| < n_2 < n_3$. If $r = m = 1$, then $n_1 < 0$ and $n_3 = |n_1| + n_2$.*

Proof. These assumptions imply that there is an integer t with $tn_2 \equiv tn_1 \equiv 1 \pmod{n_3}$. It follows that $\gcd(t, n_3) = 1$ and $t(n_1 - n_2) \equiv 0 \pmod{n_3}$. Hence n_3 divides $(n_2 - n_1)$. This is not possible if $n_1 > 0$ and can only occur with $n_1 < 0$ if $n_2 - n_1 = n_3$, i.e., $n_3 = n_2 + |n_1|$ since $|n_1| + n_2 < 2n_3$. \square

Theorem 2. *Let $S = \{n_1, n_2, n_3\}$ where $n_j \neq 0$. If $\{n_1, n_2, n_3\} \neq \{-n, n, 2n\}$ for some integer n , then $\alpha(S) \leq 5/16$.*

Proof. When the $|n_j|$'s are not distinct, [Hare and Ramsey] proves that the angular Kronecker constant is at most $3/10$. Thus there is no loss of generality in assuming $0 < |n_1| < n_2 < n_3$.

Assume $m = \gcd(n_2, n_3)$ and that the lattice is generated by $(1/m, 0)$, $(r/n_2, m/n_3)$ where we choose $-n_3/2m < r \leq n_3/2m$. If $r = 0$ we are in the rectangular lattice case developed in [Hare and Ramsey], for which the angular Kronecker constant is at most $1/5$.

We can assume $r > 0$ by replacing n_1 by $-n_1$, if necessary. Of course, $m \leq n_3/2$.

According to the previous theorem it will be enough to prove that

$$\frac{E_1 (2 |n_1| n_2 + n_3 (|n_1| + n_2))}{n_3 (|n_1| + n_2)} \leq \frac{5}{16}$$

where

$$E_1 = \max \left(\frac{m}{2(n_2 + n_3)}, \frac{r}{2(|n_1| + n_3)}, \frac{n_3 + 2rm}{2(r(n_2 + n_3) + m(|n_1| + n_3))} \right).$$

As $r, m \leq n_3/2$, if $E_1 = m/(2(n_2 + n_3))$ or $r/(2(|n_1| + n_3))$, then $E_1 \leq n_3/(4(|n_1| + n_3))$ and calling upon Lemma 1(i) gives the desired result.

Otherwise,

$$E_1 = \frac{n_3 + 2rm}{2(r(n_2 + n_3) + m(|n_1| + n_3))}.$$

Since $r \leq n_3/2m$,

$$E_1 \leq \frac{n_3}{r(n_2 + n_3) + m(|n_1| + n_3)}$$

and thus Lemma 1(ii) gives the bound if $r + m \geq 5$.

So suppose $r + m \leq 4$ and hence $\max(r, m) \leq 3$. If $n_3 \geq 12$, $\max(r, m) \leq n_3/4$, thus

$$E_1 \leq \frac{n_3 + n_3 \min(r, m)/2}{2(r(n_2 + n_3) + m(|n_1| + n_3))} = \frac{n_3(2 + \min(r, m))}{4(r(n_2 + n_3) + m(|n_1| + n_3))}.$$

and if $(r, m) \neq (1, 1)$ we can appeal to Lemma 1(iii)

From Lemma 2 we know that the case $r = m = 1$ can only arise if $\{|n_1|, n_2, n_3\}$ is a sum set. Then $E_1 \leq (n_3 + 2)/6n_3$. Moreover, $|n_1| n_2 \leq n_3^2/4$, so for $n_3 \geq 12$,

$$\frac{E_1 (2 |n_1| n_2 + n_3 (|n_1| + n_2))}{n_3 (|n_1| + n_2)} \leq \left(\frac{1}{6} + \frac{1}{3n_3} \right) \frac{3}{2} \leq \frac{7}{24} < \frac{5}{16}.$$

For the three element sets with $n_3 < 12$, we apply our computer algorithm. The greatest angular Kronecker constant is $1/4$, occuring with the set $\{1, 2, 3\}$ (and its multiples). \square

Remark 1. We conjecture that $\alpha\{n_1, n_2, n_3\} \leq 1/4$ for all three element sets other than $\{-n, n, 2n\}$. In [Hare and Ramsey] this was proved for the rectangular lattice case and for sum sets. As well, we have run our computer algorithm on all three element sets of positive integers with $n_3 \leq 50$ and the greatest Kronecker constant is $1/4$, occuring only on the integer multiples of $\{1, 2, 3\}$.

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